Period 3, Sept 16, 2024

5. Find an equation to the Tangent Line AND Normal Line to the function $h(x) = \sqrt{x}$ at $x = 4$

Tangent Line: ______________________

Normal Line: ______________________

$h(x) = \sqrt{x}$

$h(x+c) = \sqrt{x+c}$

$h'(x) = \lim_{c \to 0} \frac{h(x+c) - h(x)}{c}$

$= \lim_{c \to 0} \frac{(\sqrt{x+c} - \sqrt{x})}{c} \cdot \frac{(\sqrt{x+c} + \sqrt{x})}{(\sqrt{x+c} + \sqrt{x})}$

$= \lim_{c \to 0} \frac{\cancel{x}+c + \cancel{\sqrt{x}\sqrt{x+c}} - \cancel{\sqrt{x}\sqrt{x+c}} - \cancel{x}}{c(\sqrt{x+c} + \sqrt{x})}$

$= \lim_{c \to 0} \frac{\cancel{c}}{\cancel{c}(\sqrt{x+c} + \sqrt{x})} = \lim_{c \to 0} \frac{1}{\sqrt{x+c} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}}$

$\frac{1}{\sqrt{x} + \sqrt{x}}$

$h'(x) = \frac{1}{2\sqrt{x}}$

Tangent Line at $x = 4$

$h(4) = \sqrt{4} = 2$

Point $(4, 2)$

Slope $= \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}$

$y = mx + b$

$y = \frac{1}{4}x + b$

$2 = \frac{1}{4}(4) + b$

$2 = 1 + b$

$1 = b$

$\boxed{\text{Tangent Line } y = \frac{1}{4}x + 1}$

Normal Line $m = -\frac{4}{1} = -4$

Point $(4, 2)$

$y = -4x + b$

$2 = -4 \cdot 4 + b \Rightarrow 2 = -16 + b \Rightarrow b = 18$

Normal Line $\Rightarrow y = -4x + 18$

1. Find the ~~average rate of change~~ slope to the following in the given interval

a) $f(x) = e^x$ over the interval $[-2, 0]$

b)

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$(-2, e^{-2}), (0, e^0)$ $e^0 = 1$

$x_2 \; y_2$

$(0, 1)$

$x_1 \; y_1$

$m = \frac{e^{-2} - 1}{-2 - 0} = \frac{\frac{1}{e^2} - \frac{e^2}{e^2}}{-2} = \frac{\frac{1 - e^2}{e^2}}{-2/1} = \frac{1 - e^2}{e^2} \cdot \frac{1}{-2} = \frac{1 - e^2}{-2e^2} = \frac{e^2 - 1}{2e^2}$

$$\lim_{x \to c} \frac{F(x)-F(c)}{x-c}$$

3. Find the instantaneous rate of change of $f(x) = x^2 - 2x$ at the point $(2, 0)$, using the *alternate limit process*

$$\lim_{h \to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h \to 0} \frac{x^2+2xh+h^2-2x-2h-[x^2-2x]}{h}$$

$F(x)=x^2-2x$

$F(x+h)=(x+h)^2-2(x+h)=(x^2+2xh+h^2)-2x-2h$

$$\lim_{h \to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{2x}-2h-\cancel{x^2}+\cancel{2x}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(2x+h-2)}{\cancel{h}} = 2x+0-2 = 2x-2$$

$F'(x)=2x-2$

$F'(2)=2\cdot 2-2=4-2=2$

$F(2)=4-4=0$

$$\lim_{x \to 2} \frac{F(x)-F(2)}{x-2}$$

$$\lim_{x \to 2} \frac{x^2-2x-0}{x-2}$$

$$\lim_{x \to 2} \frac{x\cancel{(x-2)}}{\cancel{x-2}} = 2$$

4. Find an equation to the Tangent Line AND Normal Line to the function $f(x) = \frac{1}{x^2}$ *at* $x = 2$

$F(x)=\frac{1}{x^2}$

$F(x+h)=\frac{1}{(x+h)^2}$

$(x+h)^2=x^2+2xh+h^2$

$$\lim_{h \to 0} \frac{\frac{x^2}{x^2}\frac{1}{(x+h)^2} - \frac{1}{x^2}\frac{(x+h)^2}{(x+h)^2}}{h}$$

$$\lim_{h \to 0} \frac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{x^2(x+h)^2}}{\frac{h}{1}}$$

$$\lim_{h \to 0} \frac{\cancel{h}(-2x-h)}{x^2(x+h)^2}\cdot\frac{1}{\cancel{h}} = \frac{-2x-0}{x^2(x+0)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

6. At what point is the tangent line to $f(x) = x^2 - 6x + 1$ *horizontal*? $m=0$

Find $F'(x)$

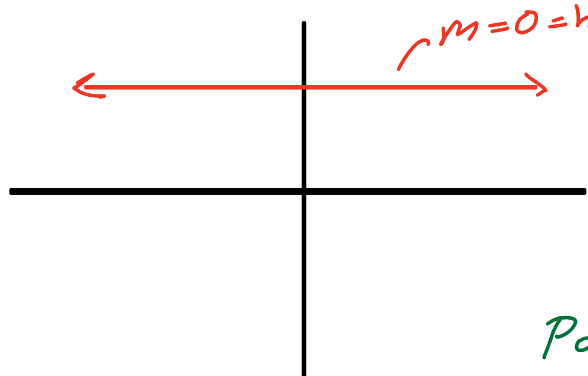
$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 6(x+h)+1 - [x^2 - 6x + 1]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{6x} - 6h \cancel{+1} - \cancel{x^2} \cancel{+6x} \cancel{-1}}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 6h}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(2x+h-6)}{\cancel{h}} = 2\cdot x + 0 - 6 = 2x - 6$$

$F'(x) = 2x - 6$ = Slope of all TangenT Lines

$m = 0$ = horizonTal TangenT



$F'(x) = 0$

$2x - 6 = 0$

$x = 3$

PoinT

$F(x) = x^2 - 6x + 1$

$F(3) = 3^2 - 6(3) + 1 = -8$

PoinT $(3, -8)$

$F(x) = 107x^{73} - 25{,}001x^{24} + 7000x^2 - 42$

Power rule

1 — $(x+y)^0 = 1$

$1 \quad 1$ — $(x+y)^1$

$1 \quad 2 \quad 1$ — $(x+y)^2$

$1 \quad 3 \quad 3 \quad 1$ — $(x+y)^3$

$1 \quad 4 \quad 6 \quad 4 \quad 1$ — $(x+y)^4$

$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$ — $(x+y)^5$

$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$ — $(x+y)^6$

$y = 3x^4$ Find $\frac{dy}{dx}$

$F(x) = 3x^4$

$F(x+h) = 3(x+h)^4$

$$\lim_{h \to 0} \frac{3(1x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + 1h^4) - 3x^4}{h}$$

$$\lim_{h \to 0} \frac{\cancel{3x^4} + 12x^3h + 18x^2h^2 + 12xh^3 + 3h^4 \cancel{-3x^4}}{h}$$

$$\lim_{h \to 0} \frac{12x^3h + 18x^2h^2 + 12xh^3 + 3h^4}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(12x^3 + 18x^2h + 12xh^2 + 3h^3)}{\cancel{h}}$$

$$\lim_{h \to 0} 12x^3 + 18x^2h + 12xh^2 + 3h^3 = 12x^3 \cancel{+ 18x^2 \cdot 0} \cancel{+ 12x \cdot 0^2} \cancel{+ 3(0)^3}$$

$$F'(x) = \frac{dy}{dx} = 12x^3$$

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

$$= \frac{4!}{0!\,4!}a^4 + \frac{4!}{1!\,3!}a^3b + \frac{4!}{2!\,2!}a^2b^2 + \frac{4!}{3!\,1!}ab^3 + \frac{4!}{4!\,0!}b^4$$

$$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$0! = 1$

$$(x+h)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3h + \binom{4}{2}x^2h^2 + \binom{4}{3}xh^3 + \binom{4}{4}h^4$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$${}_4C_0 = \binom{4}{0} = \frac{4!}{0!(4-0)!} = \frac{\cancel{4\cdot3\cdot2\cdot1}}{1\cdot\cancel{4\cdot3\cdot2\cdot1}} = 1$$

$${}_4C_1 = \binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4\cdot\cancel{3\cdot2\cdot1}}{1\cdot\cancel{3\cdot2\cdot1}} = 4$$

$${}_4C_2\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot3\cdot\cancel{2\cdot1}}{2\cdot1\cdot\cancel{2\cdot1}} = \frac{12}{2} = 6$$

$$n! = n(n-1)(n-2)(n-3)\cdots3\cdot2\cdot1$$

$$(n-1)! = (n-1)(n-2)(n-3)\cdots3\cdot2\cdot1$$

Proof Find $F'(x) = ?$

$F(x) = x^n$

$F(x+h) = (x+h)^n$

$$\lim_{h\to0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to0}\frac{\overbrace{\binom{n}{0}}^{=1}x^n + \overbrace{\binom{n}{1}}^{=n}x^{n-1}\cdot h^1 + \overbrace{\binom{n}{2}}^{=\frac{n(n-1)}{2}}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots + \overbrace{\binom{n}{n}}^{=1}h^n - x^n}{h}$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1\cdot n!} = \frac{\cancel{n(n-1)(n-2)\cdots3\cdot2\cdot1}}{1\cdot\cancel{n(n-1)(n-2)\cdots3\cdot2\cdot1}} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cdot\cancel{(n-1)(n-2)(n-3)\cdots3\cdot2\cdot1}}{1\cdot\cancel{(n-1)(n-2)(n-3)\cdots3\cdot2\cdot1}} = \frac{n}{1} = n$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)\cancel{(n-2)(n-3)(n-4)\cdots 3\cdot 2\cdot 1}}{2\cancel{(n-2)(n-3)(n-4)\cdots 3\cdot 2\cdot 1}} = \frac{n(n-1)}{2}$$

$$\binom{n}{n} = 1$$

$$\lim_{h\to 0} \frac{\cancel{1x^n} + n x^{n-1}\cdot h + \frac{n(n-1)}{2}\cdot x^{n-2}\cdot h^2 + \binom{n}{3} x^{n-3}\cdot h^3 + \cdots 1\cdot h^n \cancel{-1x^n}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}\left(n x^{n-1} + \frac{n(n-1)}{2} x^{n-2}\cdot h + \binom{n}{3} x^{n-3} h^2 + \cdots 1 h^{n-1}\right)}{\cancel{h}}$$

$$n x^{n-1} + \cancel{\frac{n(n-1)}{2} x^{n-2}\cdot 0} + \cancel{\binom{n}{3} x^{n-3}\cdot 0^2 + \cdots 0^{n-1}}$$

$$\left.\begin{aligned} F'(x) &= n x^{n-1} \\ F(x) &= x^n \end{aligned}\right\}\ \text{power rule}$$

$$F(x) = 12x^5 + 7x^2 - 4x + 9 - 5\sqrt{x}$$

$$F(x) = 12x^5 + 7x^2 - 4x^1 + 9\cdot x^0 - 5x^{\frac{1}{2}}$$

$$F'(x) = 12\cdot 5\cdot x^{5-1} + 7\cdot 2\cdot x^{2-1} - 4\cdot 1\cdot x^{1-1} + 9\cdot 0\cdot x^{0-1} - 5\cdot \tfrac{1}{2} x^{\frac{1}{2}-1=-\frac{1}{2}}$$

$$F'(x) = 60x^4 + 14x - 4 + 0 - \frac{5}{2}\cdot x^{-\frac{1}{2}}$$

$$= 60x^4 + 14x - 4 - \frac{5}{2\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}$$

$$F'(x) = 60x^4 + 14x - 4 - \frac{5\sqrt{x}}{2x}$$

$$P(x) = \frac{15x^7 + 12x^3 - 6x^5 + 4x^{3/2}}{x^2} = \frac{15x^7}{x^2} + \frac{12x^3}{x^2} - \frac{6x^5}{x^2} + \frac{4x^{3/2}}{x^2}$$

$$\frac{3}{2} - 2 = \frac{3}{2} - \frac{4}{2} = -\frac{1}{2}$$

$$P(x) = 15x^5 + 12x - 6x^3 + 4x^{-\frac{1}{2}}$$

$$P'(x) = 15 \cdot 5x^{5-1} + 12 \cdot 1 \cdot x^{1-1=0} - 6 \cdot 3x^{3-1} + 4 \cdot -\frac{1}{2} \cdot x^{-\frac{1}{2}-1=-\frac{3}{2}}$$

$$P'(x) = 75x^4 + 12 - 18x^2 - 2x^{-3/2} = 75x^4 + 12 - 18x^2 - \frac{2\sqrt{x}}{x\sqrt{x} \cdot \sqrt{x}}$$

$$F(x) = \sin x$$

$$F'(x) = \cos x$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$\lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right]$$

$$\lim_{h \to 0} \frac{\sin x \boxed{(\cos h - 1)}}{\boxed{h}} + \frac{\boxed{\sin h} \cdot \cos x}{\boxed{h}}$$

$$\underset{0}{} \qquad =1$$

$$\lim_{h \to 0} \sin x \cdot 0 + 1 \cdot \cos x = 0 + \cos x$$

$$\cos x$$

$$P(x) = \cos x$$

$$P'(x) = -\sin x$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$